

Brinkman flow past a sphere embedded in porous region

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Abstract. Creeping flow behavior exhibited by couple stress fluids through porous structure embedded a rigid sphere is analyzed in this paper. Brinkman's identity is applied to govern, motion of fluid in porous media. Separation of variables is employed to analyze the governing equation. The stream function governing the flow behavior is derived under the no slip boundary condition. The rigid sphere's resistance force has been evaluated and the variations of drag with different fluid parameters have been presented graphically. It reveals the insight, increasing the permeability parameter reduces the drag, while increase in couple stress viscosity coefficient enhances the drag.

Introduction

The research of creeping flow across a porous media is very essential in groundwater recharge, petroleum technology and also has many engineering applications. Several authors has been investigated bodies embedded in permeable structure and modeled using Brinkman's equation. Madasu and Sarkar [1] analyzed the motion associated with couple-stress fluid around a sphere situated within a permeable structure under slip boundary conditions. Deo and Maurya [2] explored the behavior of micro-polar motion of fluid inside a permeable cylindrical structure embedded in an infinite porous region. Ansari and Deo [3] investigated the influence of a magnetohydrodynamic (MHD) on the associated with an incompressible viscous fluid motion throughout a porous sphere located within another porous structure at low Reynolds number. Madasu and Bucha [4] conducted a study on the impact of a magnetohydrodynamics field on the flow behavior of an incompressible micro-polar fluid over a sphere embedded within a Brinkman permeable structure. Yadav and Deo [5] investigated the flow of an incompressible viscous fluid over a slightly modified porous sphere embedded within another porous medium, concluding that increased permeability produces a reduction in drag resistance. Jaiswal [6] addressed Stokes flow around a Reiner–Rivlin fluid-filled sphere placed inside micro-polar permeable structure with fluid-saturated, employing Brinkman's model and considering a uniform free-stream. Further, Jaiswal and Gupta [7] analyzed Stokes flow over a Reiner–Rivlin spherical body immersed in a porous matrix saturated with a polar fluid, also using the Brinkman formulation. Namdeo and Gupta [8] explored magnetohydrodynamic (MHD) impact on the fluid motion past a slightly deformed sphere situated in an unbounded porous material. Barman [9] presented a study on Newtonian fluid flow around an impervious sphere embedded within a permeable media. Pop and Ingham [10] employed the Brinkman model to examine flow past a spherical situated in a permeable domain. Additionally, Srinivasacharya and Murthy [11] investigated the drag experienced by an axisymmetric body placed in a saturated permeable structure. They found that if permeability parameter increased then drag coefficient also increase. Grosan et al. [12] explored the Brinkman model for viscous fluid flow throughout a spherical permeable region situated within a alternative porous media. Their findings indicated that the magnitude of shear stress grows with increasing porous medium parameters. Deo and Gupta [13] analyzed the drag experienced by a porous sphere subsequent in a second permeable domain and concluded that the resistance force remains unaffected by the porosity ratio. In a separate study, Leontev [14] applied the Brinkman equation combined with Navier-type boundary conditions to examine flow over both cylindrical



and spherical geometries in porous environments. Using the Brinkman framework, Jaiswal [15] examined the effect of the couple of parameters used in the Brinkman framework on the drag; it was observed that the two parameters: micro polar parameter and couplings parameter has opposite effects on the drag. They also determined the drag on a fluid-filled sphere with solid core and embedded in a porous structure (Ramalakshmi and Shukla [16]) have also assessed the drag. The term couple-stress fluids, which describes the effect of micro-structural effect in fluid was originally coined by Stokes [17, 18]. The motion behavior of the couple stresses fluid under the constraints of a slip boundary has been solved using analytical methods (Devakar and Sreenivasu [19]) gave the solution. In a related work, Ashmawy [20] determined the expressions of the resulting drag by analyzing the drag force on a spherical particle moving through a couple. stress fluid under slip conditions applied, and derived the expressions of the resulting drag.

In this study, the flow behavior of couple stress fluid flowing over a stationary rigid sphere embedded in a porous media is analyzed using Brinkman's model. A model that imposed on the surface of the sphere to obtain by no-slip boundary condition is a realistic representation of the physical scenario. The research provides detailed analytical formulations for the stream function, velocity components, stress factors, and the drag encountered by the sphere. The role of several important fluid parameters such as the permeability of the permeable media, the stress couple length scale, and the dynamic viscosity coefficient—is thoroughly examined. These effects are presented through comprehensive graphical illustrations to demonstrate how each parameter affects the drag force and overall flow dynamics. The theoretical outcomes of the present work are carefully aligned with known results to ensure consistency and justify the accuracy of the derived solutions. This study contributes to an enhanced understanding of fluid micro-structure interactions in porous media involving micro-structural effects.

Problem Formulation

Figure 1 presents a schematic representation of an axi-symmetric Brinkman flow involving a non-compressible fluid with couple stresses interacting with a rigid, stationary sphere whose radius is denoted by a , which is embedded in a flow is taken to be porous media. The flow is considered to be steady-state and symmetric about the axis passing through the sphere's center. At a sufficiently large distance from the sphere, the fluid maintains a uniform velocity U , which serves as the far-field boundary condition. This setup models the creeping motion of the fluid as it approaches and flows around the sphere, capturing the effects of both the porous media and the micro-structural couple stresses inherent in the fluid. Suppose (r, θ, ϕ) denotes that the spherical polar coordinate system at the sphere's center. As the flow under consideration is axisymmetric, the fluid motion does not vary with respect to ϕ .

The flow of an incompressible couple around a sphere in a permeable media is governed by modified Brinkman-equations that incorporate micro-structure effects. The continuity equation ensures mass conservations and formulated as:

$$\nabla \cdot \vec{v} = 0 \quad (1)$$

$$\frac{\mu}{k} \cdot \vec{v} + \mu \nabla \times \nabla \times \vec{v} + \eta \nabla \times \nabla \times \nabla \times \vec{v} + \nabla p = 0 \quad (2)$$

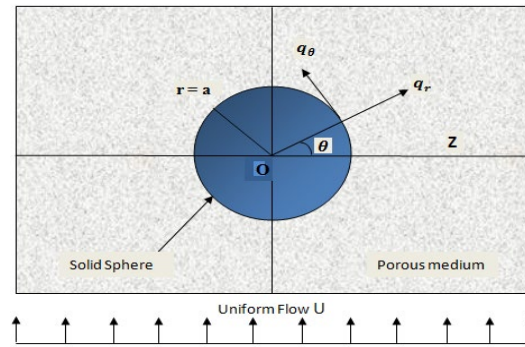


Fig.1: Flow diagram

In this analysis, the $\vec{v}$ symbol denotes the velocity vector that describes the flow field of the fluid, while p represents the pressure at any given point within the fluid domain. The parameter k corresponds to the permeability of the permeable media, which characterizes the ease with which the fluid can pass through the medium. The coefficient of dynamic viscosity of the fluid is denoted by μ , reflecting its internal resistance to shear. The parameters η and η' are stress couple viscosity coefficients, which arise due to the presence of micro-structural effects within the fluid and are associated with the rotational interactions at the micro-scale level.

To ensure the physical realism and stability of the model, these material parameters must satisfy the following set of inequalities:

$$\eta \geq \eta', \quad \eta \geq 0, \quad \mu \geq 0. \quad (3)$$

The expressions for the stress tensor and the couple stress tensor which describe the internal forces and moments within a stress fluid, are defined as follow :

$$t_{ij} = 2\mu d_{ij} - p\delta_{ij} - \frac{1}{2}e_{ijk}m_{sk,s} \quad (4)$$

$$m_{ij} = m\delta_{ij} + 4(\eta\omega_{j,i} + \eta'\omega_{i,j}) \quad (5)$$

Where $\omega_{j,i}$ is vorticity vector.

The tensor representing the deformation rate is defined by

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad (6)$$

$$\text{The vorticity vector is given by } \omega = \frac{1}{2}e_{ijk}v_{k,j} \quad (7)$$

where e_{ijk} denotes the alternating tensor and Kronecker delta is given by

$$\delta_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases} \quad (8)$$

To simplify the governing equations and facilitate analytical or numerical solution, the equations are converted into dimensionless form using appropriate dimensionless variables. The resulting dimensionless parameters are adopted

$$r = a.\tilde{r}, \vec{v} = U\tilde{v}, \nabla = \frac{1}{a}\tilde{\nabla}, p = \frac{\mu U}{a}\tilde{p} \quad (9)$$

where $\tilde{r}, \tilde{v}, \tilde{\nabla}, \tilde{p}$ are dimensionless quantities.

By replacing these variables in equations (1) and (2) and removing the bar above the dimensionless variables, we derive

$$\nabla \cdot v = 0 \quad (10)$$

$$\nabla p + \beta^2 v + \nabla \times \nabla \times v + \frac{1}{\xi^2} \nabla \times \nabla \times \nabla \times \nabla \times v = 0 \quad (11)$$

Here $\beta = \sqrt{\frac{a^2}{k}}$ and $\xi^2 = \frac{\mu a^2}{\eta}$ stands for the permeability and the microstructural length parameter in a couple stress fluid. If $\xi \rightarrow \infty$ then equation (11) converts into Brinkman's equation for Newtonian fluid.

The velocity vector in axis-symmetric flow is independent of azimuthal angle ϕ and can be express within spherical polar coordinates system (r, θ, ϕ)

$$\vec{v} = (u, v, 0) \quad (12)$$

The relation between velocity components and stream function has been given in equation (13). This stream function also satisfies the continuity equation.

$$v_r = \frac{-1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \quad (13)$$

Substituting above velocity components and eliminating the pressure from equation (10) we obtain

$$E^2(E^2 - \lambda_1^2)(E^2 - \lambda_2^2)\psi = 0 \quad (14)$$

Where $E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta}$ and $\lambda_i^2 = \frac{\xi^2 \pm \sqrt{\xi^4 - 4\beta^2}}{2}$, $i = 1, 2$.

Solution of the Problem

Separation of variables technique is adopted for solving partial differential equation (14) and the solution is denoted by

$$\psi = \frac{1}{2} \left(\frac{A}{r} + B \sqrt{r} K_{\frac{3}{2}}(\lambda_1 r) + C \sqrt{r} K_{\frac{3}{2}}(\lambda_2 r) + r^2 \right) \sin^2 \theta \quad (15)$$

Where $K_{\frac{3}{2}}(\lambda_1 r)$ and $K_{\frac{3}{2}}(\lambda_2 r)$ are known as modified Bessel functions of the second kind of fractional order $3/2$.

Putting value of stream function from equation (15) in equation (13), we have velocity components

$$v_r = -\left\{ 1 + Ar^{-3} + Br^{-\frac{3}{2}} K_{\frac{3}{2}}(\lambda_1 r) + Cr^{-\frac{3}{2}} K_{\frac{3}{2}}(\lambda_2 r) \right\} \cos \theta \quad (16)$$

$$v_\theta = \frac{1}{2} \left\{ 2 - Ar^{-3} - Br^{-\frac{3}{2}} \left(K_{\frac{3}{2}}(\lambda_1 r) + \lambda_1 r K_{\frac{1}{2}}(\lambda_1 r) \right) - Cr^{-\frac{3}{2}} \left(K_{\frac{3}{2}}(\lambda_2 r) + \lambda_2 r K_{\frac{1}{2}}(\lambda_2 r) \right) \right\} \sin \theta \quad (17)$$

The non-zero vorticity vector is

$$\omega_\theta = \frac{1}{4} \left\{ Br^{-\frac{1}{2}} \lambda_1^2 K_{\frac{3}{2}}(\lambda_1 r) + Cr^{-\frac{1}{2}} \lambda_2^2 K_{\frac{3}{2}}(\lambda_2 r) \right\} \sin \theta \quad (18)$$

The resulting expression for pressure, normal, tangential and couple stress components can be written as

$$p = \beta^2 \left(r - \frac{A}{2r^2} \right) \cos \theta \quad (19)$$

$$t_{rr} = \left[-\beta^2 \left(r - \frac{1}{2} Ar^{-2} \right) + 6Ar^{-4} + Br^{-\frac{5}{2}} \left\{ 6K_{\frac{3}{2}}(\lambda_1 r) + 2\lambda_1 r K_{\frac{1}{2}}(\lambda_1 r) \right\} + \right.$$

$$Cr^{-\frac{5}{2}} \left\{ 6K_{\frac{3}{2}}(\lambda_2 r) + 2\lambda_2 r K_{\frac{1}{2}}(\lambda_2 r) \right\} \cos \theta \quad (20)$$

$$t_{r\theta} = \frac{1}{2} 6Ar^{-4} + Br^{-\frac{5}{2}} \left\{ (6+r^2\beta^2) K_{\frac{3}{2}}(\lambda_1 r) + 2\lambda_1 r K_{\frac{1}{2}}(\lambda_1 r) \right\} + Cr^{-\frac{5}{2}} \left\{ (6+r^2\beta^2) K_{\frac{3}{2}}(\lambda_2 r) + 2\lambda_2 r K_{\frac{1}{2}}(\lambda_2 r) \right\} \sin \theta \quad (21)$$

$$m_{r\phi} = -r^{-\frac{3}{2}} \left[B \left\{ (2\eta+\eta')\lambda_1^2 K_{\frac{3}{2}}(\lambda_1 r) + \eta\lambda_1^3 r K_{\frac{1}{2}}(\lambda_1 r) \right\} + C \left\{ (2\eta+\eta')\lambda_2^2 K_{\frac{3}{2}}(\lambda_2 r) + \eta\lambda_2^3 r K_{\frac{1}{2}}(\lambda_2 r) \right\} \right] \sin \theta \quad (22)$$

$$m_{\phi r} = -r^{-\frac{3}{2}} \left[B \left\{ (\eta+2\eta')\lambda_1^2 K_{\frac{3}{2}}(\lambda_1 r) + \eta'\lambda_1^3 r K_{\frac{1}{2}}(\lambda_1 r) \right\} + C \left\{ (\eta+2\eta')\lambda_2^2 K_{\frac{3}{2}}(\lambda_2 r) + \eta'\lambda_2^3 r K_{\frac{1}{2}}(\lambda_2 r) \right\} \right] \sin \theta \quad (23)$$

$$m_{\theta\phi} = r^{-\frac{3}{2}} \left[B \left\{ (\eta-\eta')\lambda_1^2 K_{\frac{3}{2}}(\lambda_1 r) \right\} + C \left\{ (\eta-\eta')\lambda_2^2 K_{\frac{3}{2}}(\lambda_2 r) \right\} \right] \cos \theta \quad (24)$$

Boundary Conditions

The Stream function in Equation (15) consisting three arbitrary constants A, B and C. By applying the conditions below, these constants can be evaluated

$$a) \text{ The impenetrability boundary constraint on the surface of the sphere } v_r = 0 \quad (25)$$

$$b) \text{ The vanishing couple stress given by } m_{r\phi} = 0 \quad (26)$$

$$c) \text{ The no slip boundary condition } v_\theta = 0 \quad (27)$$

By applying boundary conditions, we obtain

$$A = -\frac{3 + 3\lambda_2 + \lambda_2^2}{\lambda_2^2} + \frac{3(2\eta + \eta' + 2\eta\lambda_2 + \eta'\lambda_2 + \eta\lambda_2^2)(\lambda_1^2 - \lambda_2^2 + \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2)}{\lambda_1^2\lambda_2^2\Delta_1}$$

$$B = \frac{3e^{\lambda_1}\sqrt{\frac{2}{\pi}}(2\eta+\eta'+2\eta\lambda_2+\eta'\lambda_2+\eta\lambda_2^2)}{\sqrt{\lambda_1}\Delta_1}, C = -\frac{3e^{\lambda_2}\sqrt{\frac{2}{\pi}}(2\eta+\eta'+2\eta\lambda_1+\eta'\lambda_1+\eta\lambda_1^2)}{\sqrt{\lambda_2}\Delta_1}$$

$$\text{Where } \Delta_1 = (-\lambda_1 + \lambda_2)(2\eta + \eta' + \eta\lambda_1 + \eta\lambda_2)$$

Drag Force

The drag force exerted on the rigid sphere within a permeable region is assessed using the expression

$$F_z = 2\pi a^2 \int_0^\pi r^2 \{ t_{rr} \cos \cos \theta - t_{r\theta} \sin \sin \theta \}_{r=a} \sin \sin \theta d\theta \quad (28)$$

By substituting the stress terms from equations (20) and (21) into the above equation, we derive the final form of the drag

$$F_z = \frac{2\pi\mu a U \beta^2 \left[A - 2a^3 - 2a\sqrt{a}K_{\frac{3}{2}}(\lambda_1 a)B - 2a\sqrt{a}K_{\frac{3}{2}}(\lambda_2 a)C \right]}{3} \quad (29)$$

By inserting the evaluated constants A, B and C in drag force (29), we find

$$F_z = - \frac{2\pi U \beta^2 \mu (\eta \lambda_1^3 \Delta_2 + \lambda_1^2 \Delta_2 (2\eta + \eta \lambda_2 + \eta') + 3\lambda_2 \Delta_3 + 3\lambda_1 (1 + \lambda_2) \Delta_3)}{\lambda_1^2 \lambda_2^2 (2\eta + \eta \lambda_1 + \eta \lambda_2 + \eta')}$$

Where $\Delta_2 = (3 + 3\lambda_2 + \lambda_2^2)$ and $\Delta_3 = (2\eta + \eta \lambda_2^2 + \eta' + \lambda_2(2\eta + \eta'))$

Special Case

If viscosity parameters tends to 0 i.e. $\eta \rightarrow 0$ and $a = 1$ Then the stress couple effect of fluid converted in the Newtonian flow and the drag force is $F_z = -2\pi\mu a U(3 + 3\beta + \beta^2)$ (30)

This obtained expression aligns with the result previously derived by P Madasu and P Sarkar [1].

Results and discussions

Figure 2 presents the dependence of normalized drag D_N on permeability parameter k for distinct magnitudes of coefficient of couple stress viscosity η . The results imply that enhances in permeability reduces in normalized drag D_N , whereas a rise in η causes drag to increase.

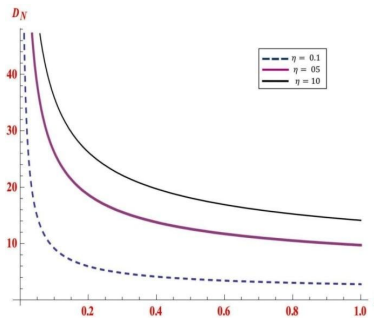


Fig.2: Behavior of drag coefficient D_N under varying parameter of permeability k .

Figure 3 displays how the normalized drag D_N changes as a function due to couple stress viscosity parameter η across a range of permeability values k . The plot clearly shows that as η increases, D_N also increases, whereas increasing the permeability k has the opposite effect, causing D_N to decrease.

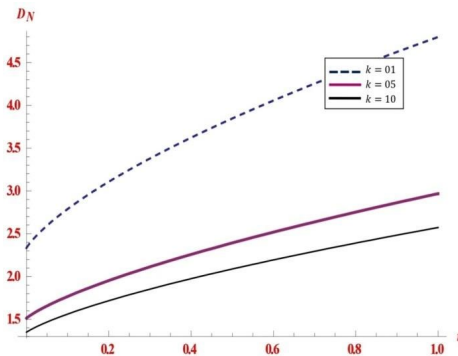


Fig.3: Variation between normalized drag and couple stress coefficient η

Fig. 4 depicts how normalized drag D_N varies with dynamic viscosity coefficient μ of the fluid over a range of permeability parameter k . The plot demonstrates that increasing the dynamics viscosity leads to a decrease in the normalized drag D_N . Furthermore, increases in permeability k consistently results in a reduction in D_N , indicating that both parameter contribute to drag reduction.

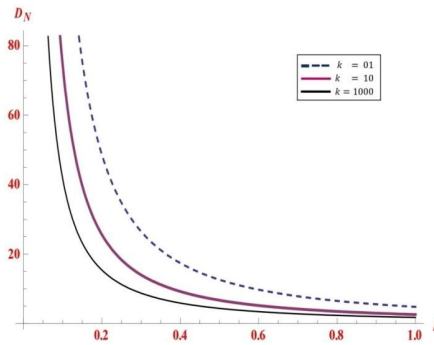


Fig.4: Variation between drag coefficient D_N and coefficient of dynamic viscosity μ .

Figure 5 demonstrates how the normalized drag D_N responds to changes in the parameter of permeability k for various inputs of the dynamic viscosity coefficient μ of fluid. The results show that greater permeability reduces drag D_N , and the drag further decreases as the viscosity of the fluid increases.

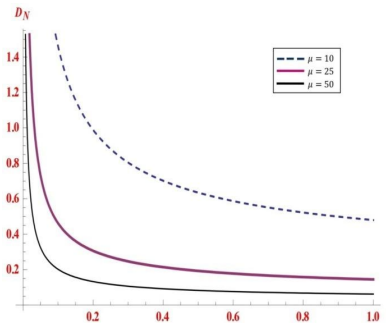


Fig.5: Graph illustrate drag behavior influenced by permeability parameter k .

Conclusions

The hydrodynamic force imposed on a rigid spherical object situated within a porous media due to the slow viscous dynamic of couple stress fluid, has been analytically determined. In addition, the flow function, components of velocity, pressure distribution, normal and tangential stresses, as well as the components of stress components has been computed for the flow field. The effect with the permeability parameter, stress couple viscosity coefficient and dynamic viscosity coefficient on the drag coefficient is illustrated through graphical representations, which clearly show the nature of their effects and noted that-

- i) For varying couple stress parameters η , rise in the permeability k results in an evident reduction in normalized drag D_N .
- ii) The drag coefficient D_N rises as micro-polar viscosity coefficient η increases, across a range of inputs of the permeability parameter k .
- iii) The increasing value of permeability parameter k reduces in coefficient drag D_N for the different inputs coefficient of dynamic viscosity η .

List of symbols

$\vec{v}$ Fluid velocity

p Pressure

k Permeability parameter

μ Coefficient of dynamic viscosity of fluid.

η and η' couple stress viscosity parameters.

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